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Steady capillary-gravity waves on the interface of a two-layer fluid over an obstruction-forced modified K-dV equation

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Abstract. The objective of this paper is to study two-dimensional steady capillary-gravity waves on the interface between two immiscible, inviscid and incompressible fluids of constant but different densities bounded by two horizontal rigid boundaries with small symmetric obstructions of compact support at lower and upper boundaries. The derivation of the forced K-dV equation, which has been extensively studied in the literature for a single-layer fluid, fails for a two-layer fluid when the density ratio of the two-layer fluid is near the square of the depth ratio. By a unified asymptotic method, a forced modified K-dV equation is derived and new types of steady solutions are discovered. Numerical results of various solutions are also presented.

1. Introduction

In this paper, we consider two-dimensional flow of two immiscible, inviscid, and incompressible fluids of different but constant densities with surface tension at the interface and bounded by two horizontal rigid boundaries. Assume that two two-dimensional objects are moving at some constant speed c along the upper and lower boundaries. A coordinate system moving with the objects is chosen so that the objects are stationary and the speed of the two-layer fluid is c far upstream. Now the problem is reduced to the steady flow of a two-layer fluid past obstructions at the boundaries.

First let us briefly review the steady flow of a one-layer fluid under gravity past an obstruction in the absence of surface tension. Numerical studies of steady flow past a semi-circular obstruction were carried out by Forbes and Schwartz [1], Vanden-Broeck [2], and Forbes [3]. They discovered two critical values F_-, F_+ of the Froude number $F = c^2/gh$, where g is the constant gravitational acceleration, h is the constant depth of the fluid far upstream, $F_- < 1$ and $F_+ > 1$. For $F < F_-$, there exists only one branch of solutions of the free surface elevation $\eta(x)$, which is almost zero behind, but periodic ahead, of the obstruction. In $F_- < F < F_+$, there exists no steady flow. For $F > F_+$, there exist two symmetric solutions of the solitary-wave type. One approaches the uniform flow far upstream and the other approaches the solitary wave solution for a fluid with constant depth, as the obstruction size tends to zero. There also appears another type of solution, which is a hydraulic fall connecting two constant states at $x = \pm\infty$. An asymptotic theory for small-amplitude steady flow past an obstruction has been developed by Shen, Shen and Sun [4] and Shen [5] and the model equation governing the flow is the Forced Korteweg–de Vries equation (FK-dV)

$$m_0'\eta_x + m_1'\eta\eta_x + m_2'\eta_{xxx} = b_x(x) ,$$

where x is the horizontal distance, m_0', m_1', m_2' are constants and $z = b(x)$ is the equation of the obstruction. Many results obtained before can be recovered by the solutions of the FK-dV. Justification of the asymptotic theory has been given by Mielke [6], Shen [7] and Sun and Shen [8].

For the problem considered here, a FK-dV can also be derived. However, besides the new Froude number F, we have three more parameters in our problem, the depth ratio h, the density ratio ρ and the Bond number τ, a nondimensional surface tension coefficient. When $\rho = h^2$, m_1' vanishes and the derivation of the FK-dV fails. To overcome this difficulty, a unified asymptotic method is developed and a forced modified K-dV equation (FMK-dV)

$$m_0 \eta_x + m_1 \eta^2 \eta_x + m_2 \eta_{xxx} = b_x(x) ,$$

is obtained if we assume that the upper obstruction is zero and lower obstruction is $z = b(x)$ which is a positive symmetric function of x with compact support. The objective of this paper is to study possible solutions of the FMK-dV.

This paper is organized as follows. In Section 2 we formulate the problem and derive a FMK-dV equation by means of a unified asymptotic method. In Section 3, there are two sections. In Section 3.1 we study the case $\tau > \tau_0$ but not near τ_0 and consider the subcritical and supercritical cases separately. Several existence theorems are proved and numerical results are also presented. In Section 3.2 we assume $\tau < \tau_0$ but not near τ_0, and study the subcritical case. An existence theorem is proved and numerical results are given. In Section 4, a detailed summary of our results is given.

2. Derivation of the modified K-dV equation with forcing terms – FMK-dV

We consider an irrotational flow of two immiscible, inviscid and incompressible fluids of different but constant densities $\rho^{*\pm}$, where $\pm$ correspond to the upper and lower fluid respectively, with surface tension T^* at the interface and bounded by two rigid boundaries. Since a two-dimensional object at the boundaries is moving with a constant speed c, we choose a coordinate system moving with the object so that in reference to the coordinate system, the object is stationary and the fluid flow becomes steady with the speed c of the two-layer fluid far upstream (Fig. 1). Then the equations of motion and boundary conditions are:

in $\Omega^{*\pm}$,

$$u^{*\pm}_{x^*} + w^{*\pm}_{z^*} = 0 , \tag{1}$$

$$u^{*\pm} u^{*\pm}_{x^*} + w^{*\pm} u^{*\pm}_{z^*} = -p^{*\pm}_{x^*} / \rho^{*\pm} , \tag{2}$$

$$u^{*\pm} w^{*\pm}_{x^*} + w^{*\pm} w^{*\pm}_{z^*} = -p^{*\pm}_{z^*} / \rho^{*\pm} - g ; \tag{3}$$

at the interface $z^* = \eta^*$,

$$u^{*\pm} \eta^*_{x^*} - w^{*\pm} = 0 , \tag{4}$$

$$p^{*+} - p^{*-} = T^* \eta^*_{x^* x^*} / (1 + (\eta^*_{x^*})^2)^{3/2} ; \tag{5}$$

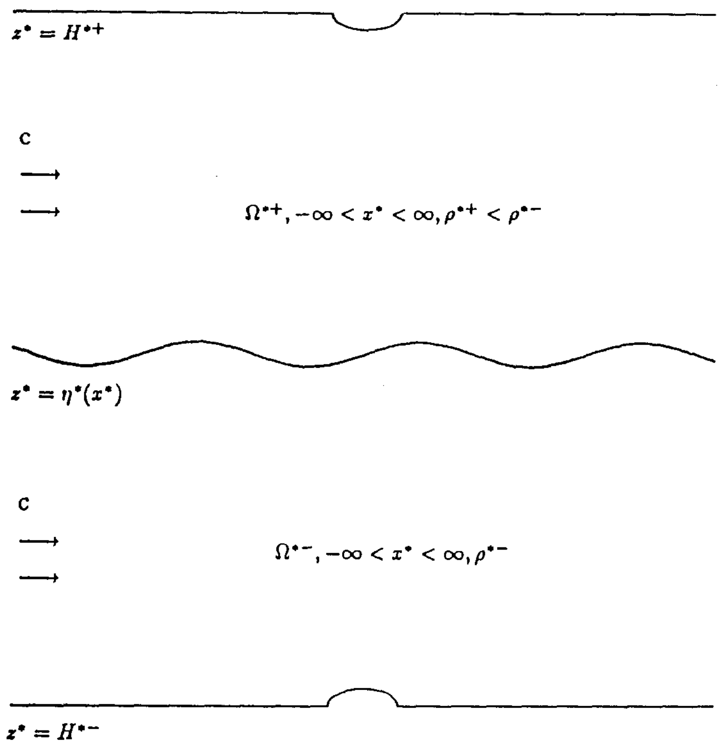

Fig. 1. Fluid domain.

at the rigid boundaries $z^* = H^{*\pm}(x^*)$,

$$w^{*\pm} - u^{*\pm}H_x^{*\pm} = 0\,. \tag{6}$$

Here $(u^{*\pm}, w^{*\pm})$ are velocities, $\rho^{\pm}$ are densities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant. Now we need to nondimensionalize these equations (1) through (6), and let L be the horizontal scale length and $H = |H^{*-}(-\infty)|$ be the vertical scale length by assuming that $H^{*-}(-\infty)$ exists. Assume that $\varepsilon = H/L \ll 1$, which is so-called long wave assumption, and introduce the following nondimensional variables:

$$(x, z) = (x^*/L, z^*/H), (u, w) = \frac{1}{\sqrt{g/H}}(u^*, \varepsilon^{-1}w^*)\,,$$

$$\eta = \varepsilon^{-1}\eta^*/H,\ p^{\pm} = p^{*\pm}/gH\rho^{*-},\ T = T^*/\rho^{*-}gH^2\,,$$

$$\rho^{\pm} = \rho^{\pm *}/\rho^{-*},\ h_0^{\pm} = H^{*\pm}/H\,.$$

In terms of them, (1) to (6) become:
in $\Omega^{\pm}$,

$$u_x^{\pm} + w_z^{\pm} = 0\,, \tag{7}$$

$$u^{\pm}u_x^{\pm} + w^{\pm}u_z^{\pm} = -p_x^{\pm}/\rho^{\pm}\,, \tag{8}$$

$$\varepsilon^2 u^\pm w_x^\pm + \varepsilon^2 w^\pm w_z^\pm = -p_z^\pm/\rho^\pm - 1\,; \tag{9}$$

at the interface $z = \varepsilon\eta$,

$$\varepsilon u^\pm \eta_x - w^\pm = 0\,, \tag{10}$$

$$p^+ - p^- = \varepsilon^2 T\eta_{xx}/(1 + \varepsilon^4\eta_x^2)^{3/2}\,; \tag{11}$$

at the rigid boundaries $z = h_0^\pm$,

$$w^\pm - h_{0x}^\pm = 0\,, \tag{12}$$

where $\rho^- = 1$ and $\rho^+ = \rho$. Then we let $h_0^\pm(x) = h^\pm + \varepsilon^3 b^\pm(x)$ with the conditions that $h^\pm$ are constants, $h^- = -1$, $h = h^+$, and $b^\pm(x)$ have compact supports.

In the following, we use a unified asymptotic method to derive the equation for $\eta(x)$. Assume that u, w and p possess an asymptotic expansion of the form

$$\phi(x, z, \varepsilon) \sim \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots\,,$$

with $u_0^\pm(x, z) = u_0 = \text{constant}$, $p_0^\pm = -\rho^\pm z$ and $w_0^\pm = 0$ and use the condition

$$\varepsilon u^+\eta_x - w^+ = \varepsilon u^-\eta_x - w^-\,, \tag{13}$$

at $z = \varepsilon\eta$ instead of the condition (1). Substituting the asymptotic expansions of u, w, p into (7) to (9) and (11) to (13), and comparing orders of ε, we can obtain a sequence of equations and boundary conditions for the successive approximations of the equations (7) to (9) and (11) to (13). The first order approximation is:
in $h^- < z < 0$ and $0 < z < h^+$,

$$u_{1x}^\pm + w_{1z}^\pm = 0\,, \tag{14}$$

$$u_0 u_{1x}^\pm = -p_{1x}^\pm/\rho^\pm\,, \tag{15}$$

$$p_{1z}^\pm = 0\,, \tag{16}$$

at $z = 0$,

$$w_1^+ - w_1^- = 0\,, \tag{17}$$

$$p_1^+ - p_1^- + \eta(p_{0z}^+ - p_{0z}^-) = 0\,, \tag{18}$$

at $z = h^\pm$,

$$w_1^\pm = 0\,. \tag{19}$$

(16) implies that $p_1^\pm$ are functions of x only. We express $p_1^\pm = f^\pm(x)$, and from (14), (15), and (19), it follows that

$$w_1^{\pm} = (z - h^{\pm}) f_x^{\pm}(x) / (u_0 \rho^{\pm}) . \tag{20}$$

From (17) and (18), and using that fact that $\eta|_{x=-\infty} = 0$, $u_1^{\pm}|_{x=-\infty} = \lambda_1$, we obtain

$$f^- = ((h^+/(u_0\rho^+))(\rho^+ - \rho^-)/(h^-/(u_0\rho^-) - h^+/(u_0\rho^+)))\eta , \tag{21}$$

$$p_1^{\pm} = c_1^{\pm} \eta , \tag{22}$$

$$c_1^- = h^+(\rho^- - \rho^+/(\rho^+ + h^+), c_1^+ = c_1^- + (\rho^+ - \rho^-) ,$$

$$u_1^{\pm} = (-c_1^{\pm} \eta / u_0^{\pm} \rho^{\pm}) + \lambda_1 , \tag{23}$$

$$w_1^{\pm} = (z - h^{\pm}) c_1^{\pm} \eta_x / (u_0^{\pm} \rho^{\pm}) . \tag{24}$$

By using a similar method, we can find the expressions for

$$u_2^{\pm}(x, z),\ w_2^{\pm}(x, z),\ p_2^{\pm}(x, z),\ u_3^{\pm}(x, z), \text{and } w_3^{\pm}(x, z)$$

with $u_2^{\pm}(x = -\infty) = \lambda$. Note that here to find $u_i^{\pm}(x, z)$, $w_i^{\pm}(x, z)$, $p_i^{\pm}(x, z)$ for $i = 1, 2, 3$, we only use the equation (13) instead of (10). (10) holds if we let, at $z = \varepsilon\eta$,

$$\varepsilon u^- \eta_x - w^- = 0 . \tag{25}$$

From (25) and the asymptotic expansion of u^-, w^-, we have at $z = 0$,

$$\begin{aligned} &u_0 \eta_x - w_1^- + \varepsilon(u_1^{-1} \eta_x - \eta w_{1z}^{-1} - w_2^-) \\ &\quad + \varepsilon^2(u_2^- \eta_x + \eta\eta_x u_{1z}^- - w_{1zz}^- \eta^2 + \eta w_{2z}^- - w_3^-) + O(\varepsilon^3) = 0 . \end{aligned} \tag{26}$$

We call u_0 the critical speed if the zeroth order term of (26), $u_0 \eta_x - w_1^-$, vanishes. Thus

$$u_0^2 = h(1 - \rho)/(\rho + h) . \tag{27}$$

Then put u_i^-, w_i^-, $i = 1, 2, 3$ into (26) to have the following equation for $\eta(x)$,

$$\begin{aligned} &2\lambda_1 \eta_x + 3(1 - \rho)(\rho - h^2)/(u_0(\rho + h)^2)\eta\eta_x \\ &\quad + \varepsilon(A_1 \eta_x + A_2 \eta\eta_x + A_3 \eta^2 \eta_x - u_0 b_x^- - (u_0 b_x^- - u_0 b_x^+/u_0 + A_4 \eta_{xxx}) \\ &\quad = O(\varepsilon^2) , \end{aligned} \tag{28}$$

where

$$\begin{aligned} &A_1 = 2\lambda - \lambda_1^2/u_0^2 , \\ &A_2 = (5 + 3\rho(\rho - 1)(1 + h)(\rho + h)^{-2})\lambda_1 , \\ &A_3 = 6\rho(1 - \rho)(1 + h)^2/(u_0(\rho + h)^3) > 0 , \\ &A_4 = h u_0(3T/u_0^2 - (1 - \rho h))/3(\rho + h) . \end{aligned}$$

If we used the long wave approximation as discussed in [4] and [5], we would obtain the forced K-dV equation

$$2\lambda_1\eta_x + 3(1-\rho)(\rho-h^2)/(u_0(\rho+h)^2)\eta\eta_x$$
$$+ (hu_0(3T/u_0^2 - (1-\rho h))/3(\rho+h))\eta_{xxx} = u_0 b_x^- + (u_0 b_x^- - u_0 b_x^+)/u_0 .$$

The nonlinear term in this equation becomes zero when $\rho = h^2$. Thus the forced K-dV equation studied in [4] and [5] fails when ρ is near h^2. Since the case for ρ not near h^2 has been studied in [4] and [5], in the following we assume $\rho = h^2 + \varepsilon\sigma^2$ and $c = u_0 + \varepsilon^2\lambda$ which implies $\lambda_1 = 0$ and is the critical speed to have the long wave approximation. Finally we obtain the so-called forced modified K-dV equation (FMK-dV);

$$2\lambda\eta_x + A_3\eta^2\eta_x - u_0 b_x^- - (u_0 b_x^- - u_0 b_x^+)/u_0 + A_4\eta_{xxx} = 0 .$$

As observed from the above equation, if $b_x^+ = (u_0 + 1)b_x^-$, the effects due to $b^+(x)$ and $b^-(x)$ cancel each other. We assume both $b^+(x)$ and $b^-(x)$ are symmetric functions with compact support. For simplicity, we may just consider the case $b_x^+ = 0$ and $b^-(x) \neq 0$ so that the effects of $b^+(x)$ and $b^-(x)$ will not cancel each other. Then after omitting higher order terms the equation (28) becomes

$$2\lambda\eta_x + A_3\eta^2\eta_x + A_4\eta_{xxx} = Cb_x(x) \tag{29}$$

where $C = u_0 + 1$.

3. Forced modified K-dV equation

Let $\tau = T/u_0^2$. We divide this section into two parts according to $\tau > \tau_0 = (1+\rho h)/3$ and $\tau < \tau_0$.

3.1. Assume $\tau > \tau_0$ but not near τ_0 and rewrite (29) as

$$\eta_{xxx} = -a_1\eta^2\eta_x + a_2\eta_x + a_3 b_x , \tag{30}$$

where

$$a_1 = 6\rho(1-\rho)(1+h)^2/(hu_0^2(\tau-\tau_0)(\rho+h)^2) > 0 ,$$

$$a_2 = -2\lambda(\rho+h)/(hu_0(\tau-\tau_0)) ,$$

$$a_3 = (u_0+1)(\rho+h)/(hu_0(\tau-\tau_0)) > 0 .$$

Integrating (30) from $-\infty$ to x gives

$$\eta_{xx} = -a_1\eta^3/3 + a_2\eta + a_3 b(x) . \tag{31}$$

When $b(x) = 0$ and $\lambda < 0$, the equation (31) can be solved directly and a solitary wave solution of the modified K-dV equation [9] is given by

$$\eta(x) = \pm(-2\lambda u_0(\rho + h)^3/(\rho(1-\rho)(1+h)^2))^{1/2} \operatorname{sech}((-2\lambda(\rho+h)/(hu_0(\tau-\tau_0)))^{1/2}x) . \tag{32}$$

In this section, we shall assume $b(x) \not\equiv 0$ and consider two cases, $\lambda < 0$ and $\lambda > 0$.

Case 1. Subcritical case $\lambda < 0$ $(a_2 > 0)$

We look for a solution $\eta(x)$ such that $\lambda < 0$ and

$$\lim_{|x|\to\infty} \left(\frac{d}{dx}\right)^j \eta(x) = 0 \quad \text{for} \quad j = 0, 1, 2 .$$

Integrating (30) from $-\infty$ to x, it follows that

$$a_2\eta - \eta_{xx} = a_1\eta^3/3 + b_1(x) , \tag{33}$$

where $b_1(x) = -a_3 b(x)$.

It is not difficult to show that (33) has a symmetric solution which decays exponentially at $|x| = \infty$ for large a_2 analytically by using contraction mapping theorem and this solution will go to zero as $b_1(x)$ tends to zero [10].

Assuming $b(x) = 0$ for $x > x_+$ and $x < x_-$ with $x_+ > x_-$, we shall construct all solutions of the equation (33) in the following ways. First find appropriate solutions for $x > x_+$ and $x < x_-$ since in these regions $b(x) = 0$ and we can find the exact solutions of (33). Then by a matching process at $x = x_+$ and $x = x_-$ and using numerical computation, we can obtain a solution for all x.

First we attempt to find a periodic solution of (33) when $b(x) = 0$. Assume $\eta(x)$ and $\eta_x(x)$ are given at some point $x = x_0$ and $\eta(x_0) = \alpha$, $\eta_x(x_0) = \beta$. We multiply η_x to (33) and integrate the resulting equation from x_0 to $x > x_0$ to have

$$(\eta_x(x))^2 = -a_1\eta^4/6 + a_2\eta^2 + d = f(\eta) ,$$

where $d = \beta^2 + a_1\alpha^4/6 - a_2\alpha^2$. To find the solution of this equation, we consider the cases $d > 0$, $d = 0$, and $d < 0$ separately. If $d > 0$, $f(\eta)$ can be factored as

$$f(\eta) = (-a_1/6)(\eta^2 - \xi_0)(\eta^2 - \xi_1) ,$$

where,

$$\xi_0 = \frac{3a_2 + 3\sqrt{a_2^2 + 2a_1 d/3}}{a_1} ,$$

$$\xi_1 = \frac{3a_2 - 3\sqrt{a_2^2 + 2a_1 d/3}}{a_1} ,$$

with $\xi_1 < 0 < \xi_0$. Hence

$$\eta = \xi_0^{1/2} \cos\phi ,$$

where

$$\gamma(x-x_0)=\int_{\phi_0}^{\phi}(1-k^2\sin^2\theta)^{-1/2}\mathrm{d}\theta,\ \phi_0=\cos^{-1}(\alpha\xi_0^{-1/2}),$$

$$\gamma=(a_1(\xi_0-\xi_1)/6)^{1/2},\ k^2=\xi_0/(\xi_0-\xi_1)<1.$$

It is clear that $\mathrm{d}x/\mathrm{d}\phi>0$. Hence $\eta(x)$ intersects the x-axis repeatedly. Suppose $\{x_i\}$ is the set of points where $\eta(x_i)=0$ for all i and $x_0\leq x_1<x_2<x_3<\dots$. Then by assuming x_i as the corresponding point of $\phi=2n\pi+\pi/2$ for some $n\in\mathbb{Z}$,

$$\begin{aligned}\int_{x_i}^{x_{i+2}}\eta(x)\,\mathrm{d}x&=\int_{2n\pi+\pi/2}^{2n\pi+5\pi/2}(\xi_0)^{1/2}\cos\phi(\mathrm{d}x/\mathrm{d}\phi)\,\mathrm{d}\phi\\&=\int_{2n\pi+\pi/2}^{2n\pi+5\pi/2}(\xi_0)^{1/2}\gamma^{-1}\cos\phi(1-k^2\sin^2\phi)^{-1/2}\,\mathrm{d}\phi\\&=\xi_0^{1/2}\gamma^{-1}\int_0^{2\pi}\sin\phi(1-k^2\cos^2\phi)^{-1/2}\,\mathrm{d}\phi\\&=\xi_0^{1/2}\gamma^{-1}\int_{-\pi}^{\pi}\sin\phi(1-k^2\cos^2\phi)^{-1/2}\,\mathrm{d}\phi=0.\end{aligned}$$

Hence the mean value of this solution $\eta(x)$ over one period is zero. If $d=0$, (33) has a solution in the following form,

$$\eta(x)=\pm(-2\lambda u_0(\rho+h)^3/(\rho(1-\rho)(1+h)^2))^{1/2}\operatorname{sech}((-2\lambda(\rho+h)/(hu_0(\tau-\tau_0)))^{1/2}x).$$

If $d<0$, $a_2^2+2a_1d/3>0$ and $f(\eta)=0$ has two distinct roots. In this case, we multiply $4\eta^2$ to (33), to obtain

$$(v_x)^2=-2a_1v^3/3+4a_2v^2+4dv=g(v),\quad v=\eta^2. \tag{34}$$

$f(v)=0$ has three different roots

$$\xi_0=\frac{3a_2+3\sqrt{a_2^2+2a_1d/3}}{a_1},$$

$$\xi_1=\frac{3a_2-3\sqrt{a_2^2+2a_1d/3}}{a_1},$$

$$\xi_2=0,\quad \xi_0>\xi_1>\xi_2.$$

The solution of (34) can be expressed as

$$v=\xi_0\cos^2\phi+\xi_1\sin^2\phi,$$

where

$$(\gamma)^{1/2}x=\int_0^{\phi}(1-k^2\sin^2\theta)^{-1/2}\,\mathrm{d}\theta,$$

$$\gamma = 3(\xi_0 - \phi_2)/3 > 0\,, \quad k^2 = (\xi_0 - \xi_1)/(\xi_0 - \xi_2) < 1.$$

It follows that $\eta = - \pm (\xi_0 \cos^2\tau + \xi_1 \sin^2 \tau)^{1/2}$. If $a^2 + 2a_1 d/3 = 0$,

$$(v_x)^2 = -a_1 v(v - 3a_2/a_1)^2\,.$$

Therefore the only possibilities are $v \equiv 0$ or $v \equiv 3a_2/a_1$ i.e. $\eta(x) \equiv 0$ or $\eta(x) \equiv \pm(3a_2/a_1)^{1/2}$. If $a_2^2 + 2a_1 d/3 < 0$,

$$(v_x)^2 = -a_1 v(\gamma^2 + (v - \delta)^2) \text{ for some } \gamma, \delta\,.$$

Hence only $\eta \equiv 0$ is possible. Therefore we can find all solutions analytically for (33) with $b(x) = 0$. But when x approaches to $-\infty$, we assume that η tends to 0. Thus only $\eta \equiv 0$ or $d = 0$, which corresponds to a solitary wave solution, is possible.

Now we need to prove the existence of the solutions of (33) in $x_- \leq x \leq x_+$. In the following, we show that for any initial values of a solution at $x = x_-$, the solution always exists in $[x_-, x_+]$ and is a C^2-function. Thus by a numerical computation, we can always find the solution in the interval as accurately as desired. The error estimates for many numerical methods to compute the solution of an ODE require the boundedness of the solution. We state the theorem and prove the boundedness of the solution.

THEOREM. $\eta_{xx} = -a_1\eta^3/3 + a_2\eta - b_1(x)$, a_1, $a_2 > 0$, *with initial data* $\eta(x_-) = \alpha$ *and* $\eta_x(x_-) = \beta$ *has a* C^2*-solution for* $x_- \leq x \leq x_+$.

Proof. It suffices to show that η is bounded. For simplicity, we can assume $x_- = -1$, $x^+ = 1$. By multiplying η_x to the given equation and integrating it from -1 to x,

$$\begin{aligned}(\eta_x)^2 &= -(a_1/6)\eta^4(x) + a_2\eta^2(x) + (\eta_x(-1))^2 + (a_1/6)(\eta(-1))^4 \\ &\quad - a_2(\eta(-1))^2 - 2\int_{-1}^{x} nb_1(t)\eta'(t)\,\mathrm{d}t \\ &= -(a_1/6)(\eta^2 - 3a_2/a_1)^2 + \beta^2 + (a_1/6)\alpha^4 - a_2\alpha^2 + 3a_2^2/(2a_1) - 2\int_{-1}^{x} b_1(t)\eta'(t)\,\mathrm{d}t\,.\end{aligned}$$

Hence

$$(\eta_x)^2 \leq N + 2\int_{-1}^{x} |b_1(t)\eta'(t)|\,\mathrm{d}t \leq N + \int_{-1}^{x} (8|b_1(t)|^2 + |\eta'(t)|^2/8)\,\mathrm{d}t\,, \tag{35}$$

by Young's Inequality, where $N = \beta^2 + (a_1/6)\alpha^4 - a_2\alpha^2 + 3a_2^2/2a_1$ and $' = \mathrm{d}/\mathrm{d}t$. Suppose η is not bounded in $[-1, 1]$. Then there exists a point $x_0 \in [-1, 1]$ such that $|\eta| \to \infty$ as x approaches x_0. Then $x_0 > -1 + \varepsilon$ for some $\varepsilon > 0$ by the existence theorem in the theory of ordinary differential equations. Let $x_0 = \inf_\xi \{\xi \in [-1, 1] \,|\, |\lim_{x\to\xi^-} \eta(x)| = \infty\}$. Choose δ so that $-1 < \delta < x_0$. Then the solution of the given differential equation exists in $[-1, \delta]$, and by (35), $\eta_x(x)^2 \leq (1/8)(\delta + 1)\sup_{-1\leq t\leq\delta}(|\eta_x(t)|^2 + 16M^2) + N$ for $x \in [-1, \delta]$. Hence $\sup_{-1\leq x\leq\delta}|\eta_x(x)|^2 \leq 16M^2 + N/(1 - (\delta + 1)/8) < 16M^2 + 8N/7$ for every δ with $-1 < \delta < x_0$. Thus $\eta'(x)$ is bounded when $x \in [-1, x_0]$, and $\eta(x) = \alpha + \int_{-1}^{x} \eta'(x)\mathrm{d}x$ is bounded which contradicts to $|\eta(x)| \uparrow \infty$ as $x \to x_0$. Therefore, we can conclude that $\eta(x)$ is bounded in $[-1, 1]$ and the solution of the given equation exists.

We have shown that the solutions of (33) always exist for $x \in \mathbb{R}$ and are bounded. Since we assume that $\eta(-\infty)=0$, only two types of solutions, $\eta(x)=0$ and solitary solutions, can appear for $x<x_-$. In the following we use numerical computation to find various solutions of (33) when the bump $b(x)$ is given by $b(x)=R(1-x^2)^{1/2}$ for $|x|\leq 1$ and $b(x)=0$ for $|x|\geq 1$ where R is a given positive constant. We divided these solutions into symmetric soliton-like solutions and unsymmetric solutions.

(1) *Symmetric soliton-like solutions*
Let

$$\eta(x)=\pm D\,\mathrm{sech}((-2\lambda(\rho+h)/(hu_0(\tau-\tau_0)))^{1/2}(x-x_0))\,,$$
$$D=(-2\lambda u_0(\rho+h)^3/(\rho(1-\rho)(1+h)^2))^{1/2}\,, \tag{36}$$

where x_0 is a phase shift. To find a solution in $|x|\leq 1$, we need only consider (33) in $-1\leq x\leq 0$ subject to $(\eta'(x))^2=-a_1\eta^4/6+a_2\eta^2$ at $x=-1$ and $\eta'(x)=0$ at $x=0$. This problem can be solved numerically by a shooting method and the phase shift x_0 is then determined by (36) for $x=-1$. The numerical results are presented in Figs. 2 and Fig. 3. In Fig. 2 we show the relationship between $A=\eta(0)$ and λ where $R=1$. Three typical soliton-like solutions corresponding to $\lambda=-3.0$, $R=1.0$ are shown in Fig. 3.

(2) *Unsymmetric solutions*
Assume $\eta(-\infty)=0$ and η is periodic ahead of the bump. We have two choices for $x\leq -1$, one is $\eta\equiv 0$ and the other is

$$\eta(x)=\pm D\,\mathrm{sech}((-\lambda(\rho+h)/(hu_0(\tau-\tau_0)))^{1/2}(x-x_0))\,,$$
$$D=(-\lambda u_0(\rho+h)^3/(\rho(1-\rho)(1+h)^2))^{1/2}\,. \tag{37}$$

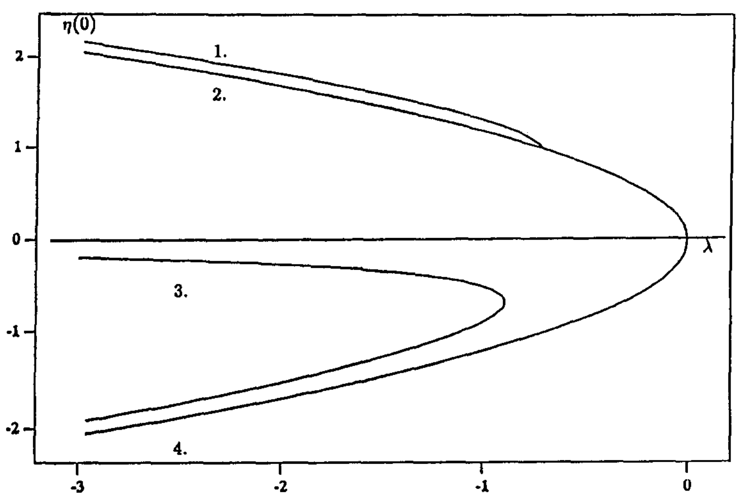

Fig. 2. Relationship between $\eta(0)$ and λ with $T=3$, $\rho=0.25$, and $h=0.5$. 1. Positive Solitonlike solutions. 2. Positive Soliton solutions without forcing. 3. Negative Solitonlike solutions. 4. Negative Solitonlike solutions without forcing.

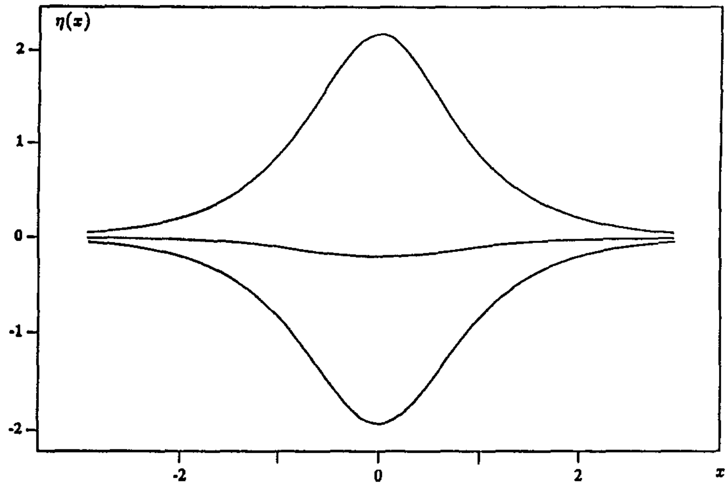

Fig. 3. Symmetric Solitonlike solutions with $T = 3$, $\lambda = -3$, $\rho = 0.25$, and $h = 0.5$.

Since we have established the existence of a solution of (33) for $[x_-, x_+]$ and derived the possible solutions of (33) in $[x_+, \infty)$ for any values of $\eta(x^+)$ and $\eta'(x^+)$, we can solve (33) by Runge-Kutta Method and use (37) or zero for $(-\infty, -1]$.

We present the numerical results of this case in Figs. 4–6. Figure 4 shows the second type solutions which have (37) as their solutions in $(-\infty, -1]$. In Fig. 5 we show the relationship between $\eta(0)$ and λ when $b(x) = (1 - x^2)^{1/2}$ for different values of $\eta(-1)$. We also consider a third type solution which is zero for $x \leqslant -1$ and periodic for $x \geqslant 1$. A typical solution corresponding to $\lambda = -2$, $b(x) = (1 - x^2)^{1/2}$ is presented in Fig. 6.

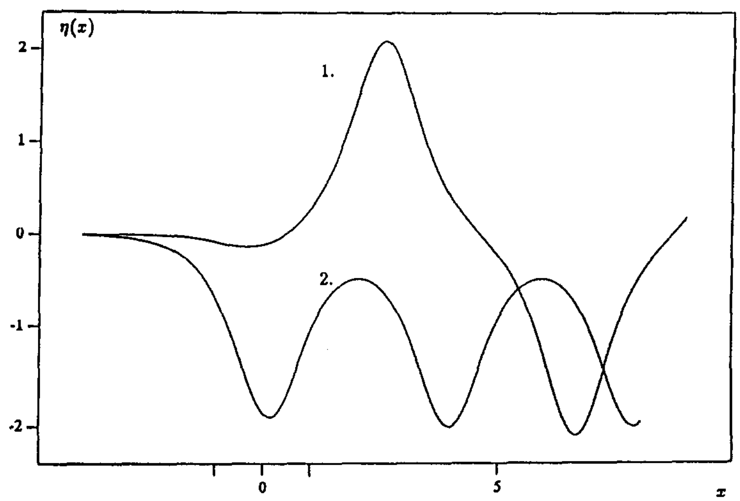

Fig. 4. Typical second type solutions with $\lambda = -3$, $T = 3$, $\rho = 0.25$, and $h = 0.5$. 1. Mean depth is 0 behind the obstruction. 2. Mean depth is not 0 behind the obstruction.

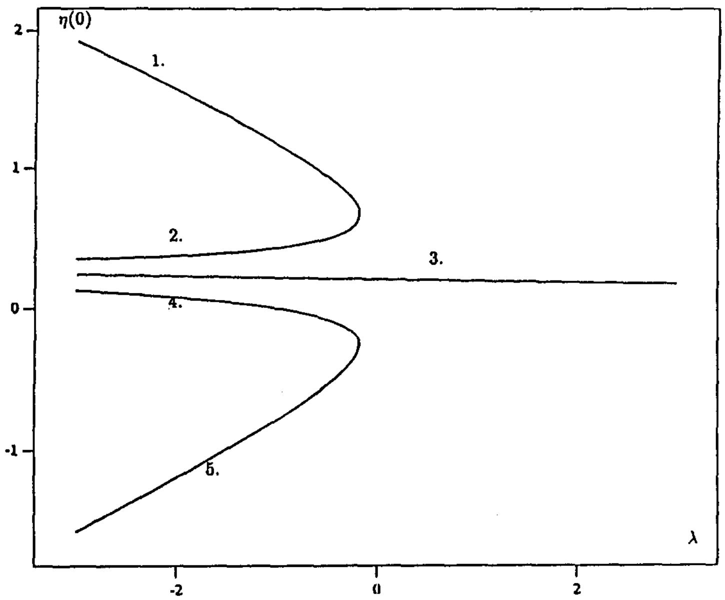

Fig. 5. Relationship between $\eta(0)$ and λ for different $\eta(-1)$ with $T=3$, $\rho=0.25$, and $h=0.5$. 1. $\eta(-1)=0.5$ and $\eta_x(x)>0$. 2. $\eta(-1)=0.5$ and $\eta_x(x)<0$. 3. $\eta(-1)=0$ and $\eta_x(x)=0$. 4. $\eta(-1)=-0.5$ and $\eta_x(x)>0$. 5. $\eta(-1)=-0.5$ and $\eta_x(x)<0$.

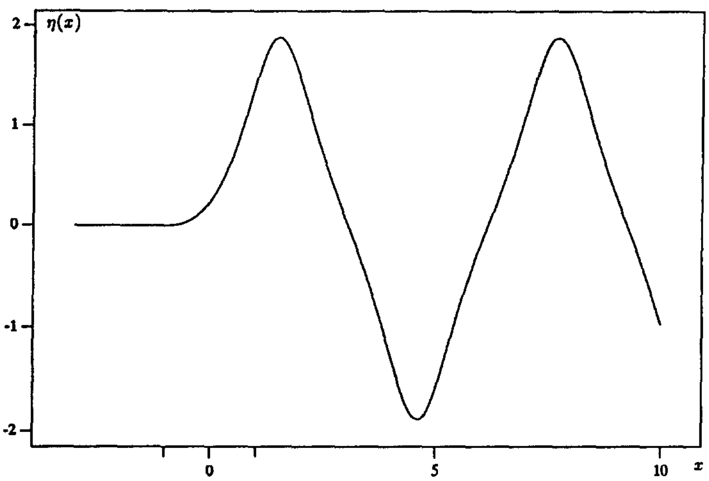

Fig. 6. Typical third type solution when $\lambda<0$ with $T=3$, $\lambda=-2$, $\rho=0.25$, and $h=0.5$.

Case 2. *Supercritical case* $\lambda>0$

Similar to the method used in case 1, we first consider homogeneous equation

$$\eta_{xx} = -a_1\eta^3/3 + a_2\eta\,, \tag{38}$$

with $a_1>0$, $a_2<0$ since $\lambda>0$. The bounded solutions of (38) can be found explicitly by using elliptic functions. We assume $x_-=-1$ and $x_+=1$. Let $\eta(1)=\alpha$ and $\eta_x(1)=\beta$. We multiply $\eta_x(x)$ to (38) and integrate the resulting equation from 1 to $x>1$ to obtain

$$(\eta_x(x))^2 = -(a_1/6)\eta^4 + a_2\eta^2 + d$$
$$= -(a_1/6)(\eta^2 - \xi_0)(\eta^2 - \xi_1)\,, \tag{39}$$

where

$$d = \beta^2 + (a_1/6)\alpha^4 - a_2\alpha^2 > 0\,,$$

$$\xi_0 = \frac{3a_2 + 3\sqrt{a_2^2 + 2a_1 d/3}}{a_1} > 0\,,$$

$$\xi_1 = \frac{3a_2 - 3\sqrt{a_2^2 + 2a_1 d/3}}{a_1} < 0\,.$$

The solution of the equation (39) is given by

$$\eta = \xi_0^{1/2} \cos\phi\,,$$

where

$$\gamma(x - x_0) = \int_{\phi_0}^{\phi} (1 - k^2 \sin^2\theta)^{-1/2}\, d\theta\,, \quad \phi_0 = \cos^{-1}(\alpha\xi_0^{-1/2})\,,$$
$$\gamma = (a_1(\xi_0 - \xi_1)/6)^{1/2}\,, \quad k^2 = \xi_0/(\xi_0 - \xi_1) < 1\,.$$

Thus for any α and β, a solution of (38) is periodic. We can show that the mean value of η over one period is zero by a similar method as in case 1. For $x < -1$, the only solution is $\eta(x) = 0$ since $\eta(-\infty) = 0$.

Also as in case 1, we can show that if $\eta(-1)$ and $\eta_x(-1)$ are given, then

$$\eta_{xx} = -a_1\eta^3/3 + a_2\eta - b_1(x)\,, \tag{40}$$

has a bounded C^2-solution in $[-1, 1]$.

Having established the existence of the solution, we can find the global solution of $\eta_{xx} = -a_1\eta^3/3 + a_2\eta - b_1(x)$ by using a shooting method to connect zero solution behind the bump and periodic solution ahead of the bump. We present the numerical result in Fig. 7, where $b(x) = (1 + x^2)^{1/2}$ for $|x| \leq 1$ and $b(x) = 0$ for $|x| > 1$, and $\lambda = 1.0$.

3.2. In this section we consider (29) for the case $\tau < \tau_0$ but τ is not near τ_0. If we let the support of $b(x)$ be in $[x_-, x_+]$ and integrate (29) once, by $\eta(-\infty) = \eta_{xx}(-\infty) = 0$ we obtain

$$\eta_{xx} = A_1\eta^3 - A_2\eta + a_3 b(x)\,, \tag{41}$$

where

$$A_1 = 6\rho(1 - \rho)(1 + h)^2/(hu_0^2(\tau_0 - \tau)(\rho + h)^2) > 0\,,$$
$$A_2 = -2\lambda(\rho + h)/(hu_0(\tau_0 - \tau))\,,$$
$$a_3 = -(u_0 + 1)(\rho + h)/(hu_0(\tau_0 - \tau) < 0\,.$$

First let us consider the case $\lambda < 0$, that is, $A_2 > 0$. When $b(x) = 0$, we consider an initial

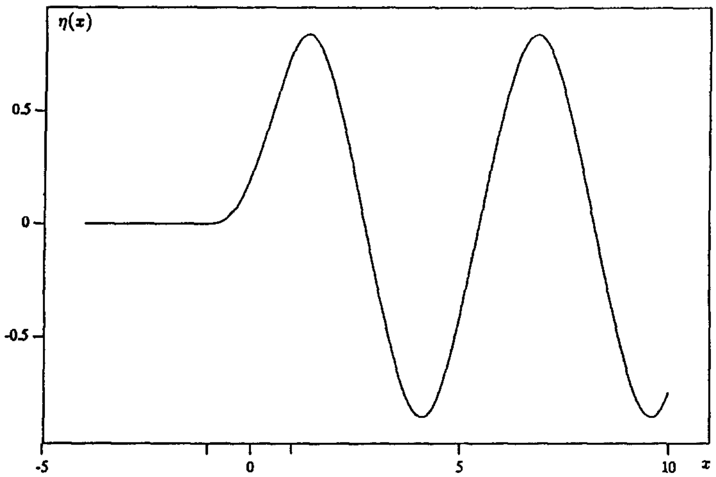

Fig. 7. Typical third type solution when $\lambda > 0$ with $T = 3$, $\lambda = 1$, $\rho = 0.25$, and $h = 0.5$.

value problem for (41) with initial value $\eta(x_+) = \alpha$, $\eta_x(x_+) = \beta$. Then by integrating it from x_+ to $x > x_+$,

$$(\eta_x(x))^2 = (A_1/2)\eta(x)^4 - A_2\eta(x)^2 + d \quad \text{where} \quad d = \beta^2 - (A_1/2)\alpha^4 + A_2\alpha^2 . \tag{42}$$

If $\alpha = \beta = 0$, then (42) has the trivial solution $\eta(x) \equiv 0$. If $A_2^2 - 2A_1 d = 0$, then (42) has a solution $\eta(x) = \pm(A_2/A_1)^{1/2} \tanh A_2 x/2$. If $A_2^2 - 2A_1 d > 0$,

$$(A_1/2)\eta(x)^4 - A_2\eta(x)^2 + d = (A_1/2)(\eta^2 - \xi_0)(\eta^2 - \xi_1)$$

where

$$\xi_0 = A_2/A_1 + (A_2^2 - 2A_1 d)^{1/2}/A_1 , \quad \xi_1 = A_2/A_1 - (A_2^2 - 2A_1 d)^{1/2}/A_1 .$$

Hence, the solution of (42) is the following: When $\xi_0 > \xi_1 > 0$, $\eta(x) = \xi_1^{1/2} \sin \phi$, where

$$(A_1\xi_0/2)^{1/2}(x - x_+) = \int_{\phi_0}^{\phi} (1 - k^2 \sin\theta)^{-1/2} d\theta ,$$

$$\phi_0 = \sin^{-1}(\alpha\xi_1^{-1/2}) , \quad k^2 = \xi_1/\xi_0 < 1 .$$

This solution $\eta(x)$ tends to 0 as $\xi_1 \to 0^+$. When $\xi_0 > 0 > \xi_1$, $\eta(x) = \xi_0^{1/2} \cos \phi$, where

$$\gamma(x - x_+) = \int_{\phi_0}^{\phi} (1 - k^2 \sin\theta)^{-1/2} d\theta , \quad \gamma = (A_1(\xi_0 - \xi_1)/2)^{1/2} ,$$

$$\phi_0 = \cos^{-1}(\alpha\xi_0^{-1/2}) , \quad k^2 = \xi_0/(\xi_0 - \xi_1) < 1 .$$

This solution $\eta(x)$ tends to $(2A_2/A_1)^{1/2} \operatorname{sech}(A_2^{1/2} x - \phi)$, where ϕ is a phase shift determined by initial values, as $\xi_1 \to 0^-$. If $A_2^2 < 2A_1 d$, $\eta_x(x) = \pm(d_1 + (A_1/2)(\eta^2(x) + c_1)^2)^{1/2}$ for some $d_1 > 0$ and the solution is unbounded.

To find the solution of (41) in $[x_-, x_+]$, we can use the contraction mapping theorem to show that if $-\lambda$ is large enough, (41) has bounded C^2-solution in $[x_-, x_+]$. Hence, by using the matching process as before, we can find the solution for all real x.

Next we consider the case for $\lambda > 0$, that is, $-A_2$ is positive in (41). Let us study the following equation:

$$\eta_{xx} = A_1\eta^3 - A_2\eta,\ A_1 > 0,\ -A_2 > 0,\ \eta(x_+) = \alpha,\ \eta_x(x_+) = \beta\ . \tag{43}$$

We integrate (43) from x_+ to $x > x_+$ to obtain

$$\eta_x^2 = (A_1/2)\eta^4 - A_2\eta^2 + d\ ,$$

where $d = \beta^2 - A_1\alpha^4/2 + A_2\alpha^2$. Also if $\eta(x_0) = \alpha_1$, $\eta_x(x_0) = \beta_1$, then $d = \beta_1^2 - A_1\alpha_1^4/2 + A_2\alpha_1^2$ for all $x_0 \geqslant x_+$. If $d > 0$, then $\eta_x^2(x) \geqslant d$ for all $x \geqslant x_+$ and hence $\eta(x)$ diverges. Also if $d \geqslant 0$, then $\eta \neq 0$ unless $\eta \equiv 0$. Therefore without loss of generality, we assume that $\eta(x) > 0$ for all $x > x_+$. By (43), $\eta_{xx}(x) > 0$. Thus $\eta_x(x)$ increases with x and $\eta_x(x) \leqslant 0$ by boundedness of $\eta(x)$. Hence $\eta_x(x) \to 0$ and $\eta_{xx}(x_n) \to 0$ for some $x_n \to \infty$, and $\eta(x) \to \text{constant} \geqslant 0$ as $x \to \infty$. Then by (43) again $\eta(x_n) \to 0$ which means $d = 0$. If $d = 0$,

$$\eta(x) = (2A_2/A_1 - (2A_2/A_1)(1 + \gamma^2 f(x))/(1 - \gamma^2 f(x))^{1/2}\ ,$$

$$f(x) = \exp(-2(-A_2)^{1/2}(x - x_+))\ ,$$

where $\gamma = ((\alpha^2 - 2A_2/A_1)^{1/2} - (-2A_2/A_1)^{1/2})/\alpha$. We can easily see $\eta(x)$ tends to 0 as $x \to \infty$ in this case. By a numerical calculation we find that if $x_- = -1$ and $x_+ = 1$ with $b(x)$ as before and $\eta(-1) = \eta_x(-1) = 0$, then d is always positive and no bounded solution exists.

The numerical results of these cases are given in Figs. 8 to 12. In Fig. 8 a hydraulic fall solution is given, which is a limiting solution of Fig. 9, where a typical periodic solution is presented. Figure 10 presents a one-hump solution which is another limiting solution of Fig. 9. Figures 11 and 12 show multi-crest solutions which take place as λ decreases.

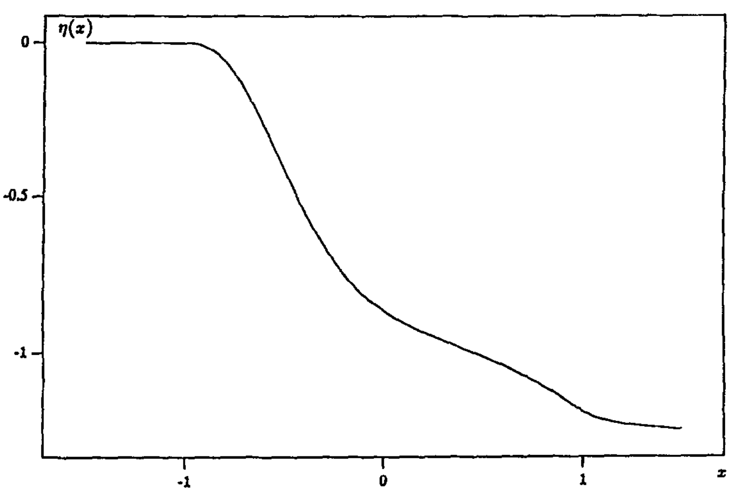

Fig. 8. Hydraulic Fall solution with $T = 10^{-3}$, $\lambda = -2.165817$, $\rho = 0.25$, and $h = 0.5$.

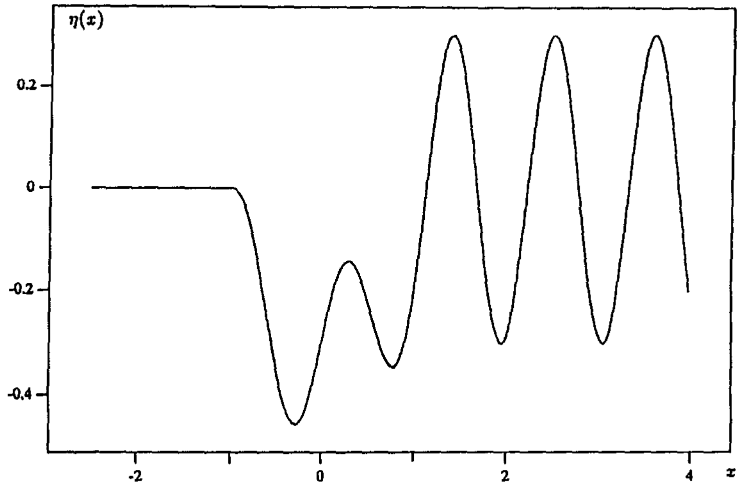

Fig. 9. Typical third type solution when $\tau < \tau_0$ with $T = 10^{-3}$, $\lambda = -3$, $\rho = 0.25$, and $h = 0.5$.

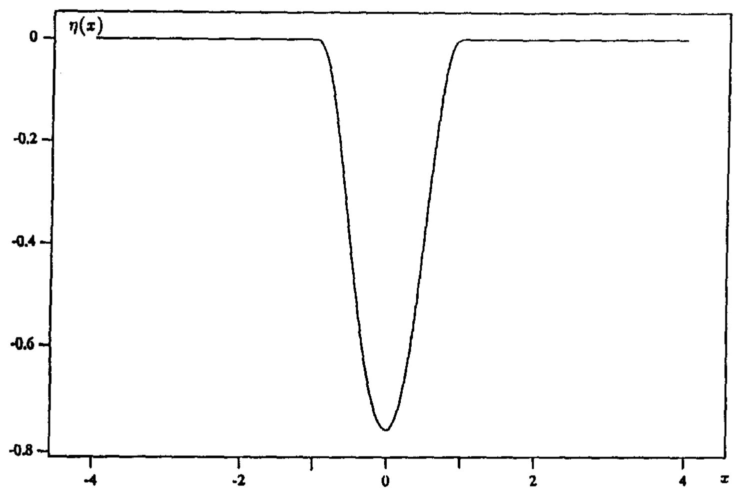

Fig. 10. Symmetric solution with one crest with $T = 10^{-3}$, $\lambda = -2.26245$, $\rho = 0.25$, and $h = 0.5$.

4. Conclusions

Our results are summarized as follows:

The forced modified K-dV equation (FMK-dV)

$$2\lambda\eta_x + 6(\rho(1-\rho)(1+h)^2/(u_0(\rho+h)^3))\eta^2\eta_x + hu_0((\tau-\tau_0)/(\rho+h))\eta_{xxx} = (u_0+1)b_x(x)\ ,$$

which describes the long wave approximations of the solutions of exact equations for a two-layer fluid in the critical cases with ρ near h^2, possesses the following possible solutions.

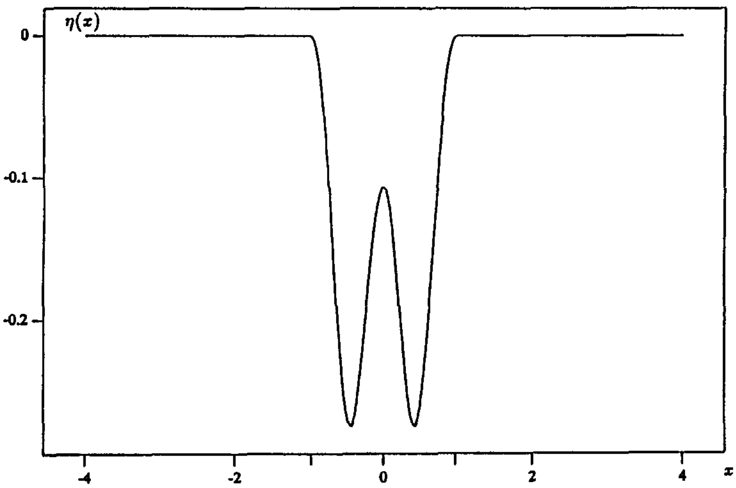

Fig. 11. Symmetric solution with two crest with $T = 10^{-3}$, $\lambda = -4.457567$, $\rho = 0.25$, and $h = 0.5$.

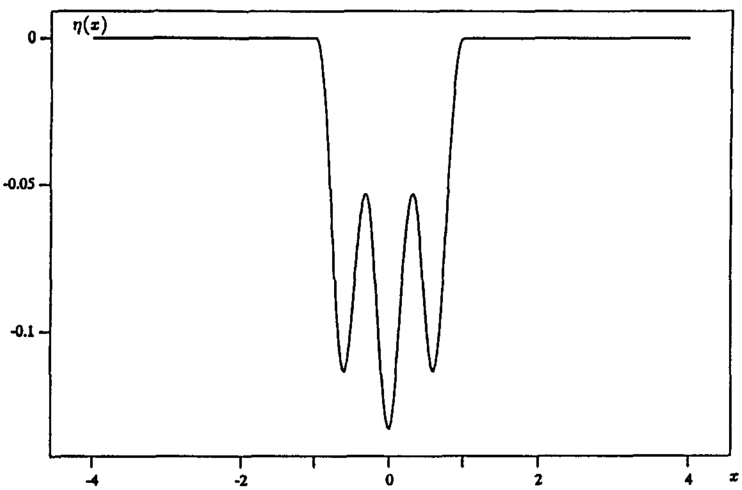

Fig. 12. Symmetric solution with three crest with $T = 10^{-3}$, $\lambda = -9.128716$, $\rho = 0.25$, and $h = 0.5$.

I. $\tau > \tau_0$

Case 1. *Subcritical case* $\lambda < 0$

There appear three types of solutions. The first type consists of symmetric solitary wave solutions which tend to zero as $|x| \to \infty$. For each λ less than some critical value, there are always three one-crest solutions, one with positive amplitude and the other two with negative amplitude. The amplitude of the positive one-crest solution is larger than that of the positive free solitary wave, while the amplitude of the negative free solitary wave is in between the amplitudes of the two remaining negative symmetric solutions. Note that the modified K-dV equation without forcing possesses two solitary wave solutions with different signs, which are called free solitary waves. There are also positive two-crest symmetric solutions. However, their numbers and amplitudes are very sensitive to the change of λ and the size of the obstruction and exhibit a rather chaotic behavior.

The second and third types consist of unsymmetric solutions. The second type unsymmetric solution possesses part of a free solitary wave behind, and a periodic solution of the MK-dV ahead of, the obstruction. For any λ less than some critical value, two solutions can always be found corresponding to the same initial value $\eta(-x_0)$ prescribed at the left endpoint $x = -x_0$ of the obstruction support. The mean value over one period of the periodic solution ahead of the obstruction is zero unless $\eta(-x_0)$ falls between the values of the two negative symmetric solutions at $x = -x_0$. The third type solutions are zero behind, and periodic ahead of, the obstruction. The mean value of the periodic part of this solution is always zero.

Case 2. *Supercritical case* $\lambda > 0$
Only third type solutions appear and the mean value of the periodic part is always zero.

II. $\tau < \tau_0$

Case 1. *Subcritical case* $\lambda < 0$
As λ decreases from $\lambda = 0$, we can only find unbounded solutions until a cut-off value of λ is reached. At the cut-off value of λ there appears a hydraulic-fall solution. As λ decreases more, the third type solutions emerge. However, we have discovered a rather surprising result that at discrete values of λ there are multi-crest solutions without a periodic part embedded in the third type solutions.

Case 2. *Supercritical case* $\lambda > 0$
There is no bounded solution for $\lambda > 0$ numerically.

In summary we consider the physical problem of steady state flow past a positive, symmetric body at the horizontal bottom of a two-layer fluid with surface tension at the free surface. The derivation of the FK-dV fails when $\rho = h^2$, and by a united approach an MFK-dV is obtained. Two parameters appear in the equation and can affect its solution behavior. One is the parameter λ, a measure of the deviation of the flow speed at far upstream from the critical speed u_0, and the other is the difference $\tau - \tau_0 = \tau^*$. Mathematically we study different types of solutions which may appear in different regions of the λ, τ^*-plane. This investigation may help us understand the flow patterns under parameter change in a two-layer fluid with obstructions at boundaries.

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